

An Effective Preconditioner Based on Schur Complement Reduction for Integral-Equation Formulations of Dielectric Problems[†]

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Introduction

We consider effective preconditioning of recently proposed two integral-equation formulations for dielectrics; the combined tangential formulation (CTF) and the electric and magnetic current combined-field integral equation (JMCFIE) [1]. These two formulations are of utmost interest since CTF yields more accurate results and JMCFIE yields better-conditioned systems than other formulations [2].

Integral-equation formulations for dielectrics are obtained by simultaneous discretization of the electric and magnetic surface currents and result in block-partitioned linear systems in the form of

$$\begin{bmatrix} \bar{\mathbf{Z}}_{11} & \bar{\mathbf{Z}}_{12} \\ \bar{\mathbf{Z}}_{21} & \bar{\mathbf{Z}}_{22} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}_J \\ \mathbf{a}_M \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}, \quad \text{or} \quad \bar{\mathbf{Z}} \cdot \mathbf{a} = \mathbf{v}, \quad (1)$$

where \mathbf{a}_J and \mathbf{a}_M are the coefficient vectors of the basis functions expanding the electric and magnetic currents, respectively, and $\mathbf{v}_{1,2}$ represent excitation vectors obtained by testing the incident fields. The iterative solutions of the resulting dense systems become feasible with the multilevel fast multipole algorithm (MLFMA). However, the solutions of such block-partitioned matrices often suffer from slow convergence, due to highly indefinite nature of resulting matrices. Hence, effective preconditioners should be applied to these systems in order to increase robustness and efficiency.

Iterative Approximate Schur Preconditioner

We use the near-field matrix for preconditioning, *i.e.*, $\bar{\mathbf{M}} = \bar{\mathbf{Z}}^{NF}$. For the solution of the preconditioning system, the Schur complement reduction method is used [3]. This method reduces the solution of the block-partitioned near-field system

$$\begin{bmatrix} \bar{\mathbf{Z}}_{11}^{NF} & \bar{\mathbf{Z}}_{12}^{NF} \\ \bar{\mathbf{Z}}_{21}^{NF} & \bar{\mathbf{Z}}_{22}^{NF} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}, \quad (2)$$

to the solution of the following two systems. First, \mathbf{y} is found using

$$\bar{\mathbf{S}} \cdot \mathbf{y} = \mathbf{g} - \bar{\mathbf{Z}}_{21}^{NF} \cdot (\bar{\mathbf{Z}}_{11}^{NF})^{-1} \cdot \mathbf{f} = \mathbf{g}', \quad (3)$$

where

$$\bar{\mathbf{S}} = \bar{\mathbf{Z}}_{22}^{NF} - \bar{\mathbf{Z}}_{21}^{NF} \cdot (\bar{\mathbf{Z}}_{11}^{NF})^{-1} \cdot \bar{\mathbf{Z}}_{12}^{NF} \quad (4)$$

is the Schur complement matrix. Then, \mathbf{x} can be computed by solving

$$\bar{\mathbf{Z}}_{11}^{NF} \cdot \mathbf{x} = \mathbf{f} - \bar{\mathbf{Z}}_{12}^{NF} \cdot \mathbf{y} = \mathbf{f}'. \quad (5)$$

Since the inversion of the sparse matrix $\bar{\mathbf{Z}}_{11}^{NF}$ is unfeasible, in the right-hand side of (3) and Schur complement matrix (4), we approximate the inverse of $\bar{\mathbf{Z}}_{11}^{NF}$ with a sparse approximate inverse

[†]This work was supported by the Scientific and Technical Research Council of Turkey (TUBITAK) under Research Grants 105E172 and 107E136, by the Turkish Academy of Sciences in the framework of the Young Scientist Award Program (LG/TUBA-GEBIP/2002-1-12), and by contracts from ASELSAN and SSM.

(SAI) [4]. Then, solutions of (3) and (5) are approximated by a few iterations of the generalized minimal residual method (GMRES). We call these preconditioning solutions inner solutions and the preconditioning scheme the iterative approximate Schur preconditioner (ASP). Note that we do not need to compute and store the Schur complement matrix \bar{S} ; we only need to provide the application of \bar{S} to a vector in each step of the inner iterative solver of (3).

We increase the performance of iterative ASP by accelerating the inner solves (3) and (5) with SAI of \bar{Z}_{22}^{NF} and \bar{Z}_{11}^{NF} , respectively. Note that for CTF and JMCIE, $\bar{Z}_{11}^{NF} = \bar{Z}_{22}^{NF}$, hence SAI of \bar{Z}_{11}^{NF} serves as a preconditioner for (3), assuming that \bar{Z}_{22}^{NF} is the dominant term in the Schur complement matrix. Therefore, SAI is used both in approximating the inverse of \bar{Z}_{11}^{NF} and to accelerate the iterative solutions of (3) and (5). In Table 1, we show the performance of the SAI preconditioner for the solutions of (3) and (5), where we compare the number of iterations obtained with the SAI-preconditioner and the no-preconditioner (No PC) cases for 10^{-6} residual error. For both formulations, we observe that SAI is very successful and decreases the iteration counts drastically. Furthermore, contrary to the no-preconditioner case, the number of iterations does not increase for the SAI preconditioner as the number of unknowns increase. In Fig. 1, we illustrate the application of iterative ASP in a step of a Krylov subspace solver. Note that, if the number of iterations of the inner solver is not fixed, the use of iterative ASP requires the outer solver to be a flexible solver [3].

Table 1: Number of iterations of the systems in (5) and (3) for the sphere problem.

Number of Unknowns	CTF				JMCIE			
	$\bar{Z}_{11}^{NF} \cdot x = f'$		$\bar{S} \cdot y = g'$		$\bar{Z}_{11}^{NF} \cdot x = f'$		$\bar{S} \cdot y = g'$	
	No PC	SAI	No PC	SAI	No PC	SAI	No PC	SAI
29,742	217	10	213	10	38	6	43	9
65,724	243	10	238	9	39	6	44	9
264,006	294	10	282	9	41	6	45	9

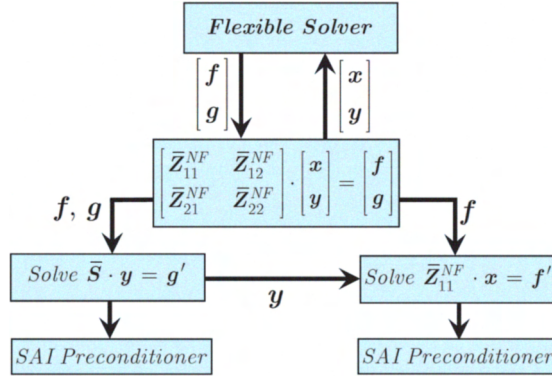


Figure 1: Illustration of the application of ASP in a step of an iterative solver.

Results

In Fig. 2, we show the total solution times (setup of SAI and iterations) obtained with the no-restart flexible GMRES for a sphere problem with a dielectric constant of 4.0 and for a real-life problem of a dielectric lens [5], which has a higher relative permittivity of 12.0. We solve problems up to 1.5 million unknowns for the sphere and up to 600,000 unknowns for the lens. Iterative ASP is compared to No PC and a four-partition block-diagonal preconditioner (4PBDP) [2], which uses the self interactions of the lowest-level clusters of MLFMA. However, this preconditioner worsens the convergence behaviour of CTF. We use 0.1 residual error and a maximum of three iterations

for the inner solutions. First, we note that these problems are much harder to solve with other non-optimal solvers, such as biconjugate gradient stabilized (BiCGStab) [2]. GMRES, on the other hand, is able to solve these problems without a preconditioner, though it may require many iterations. Using iterative ASP, both iteration counts and solution times of CTF and JMCIE are significantly improved. Moreover, for the sphere problem, iteration counts and solution times of CTF are very close to those of JMCIE. Hence, CTF is preferable to JMCIE when iterative ASP is used as the preconditioner since it produces more accurate results than JMCIE [2].

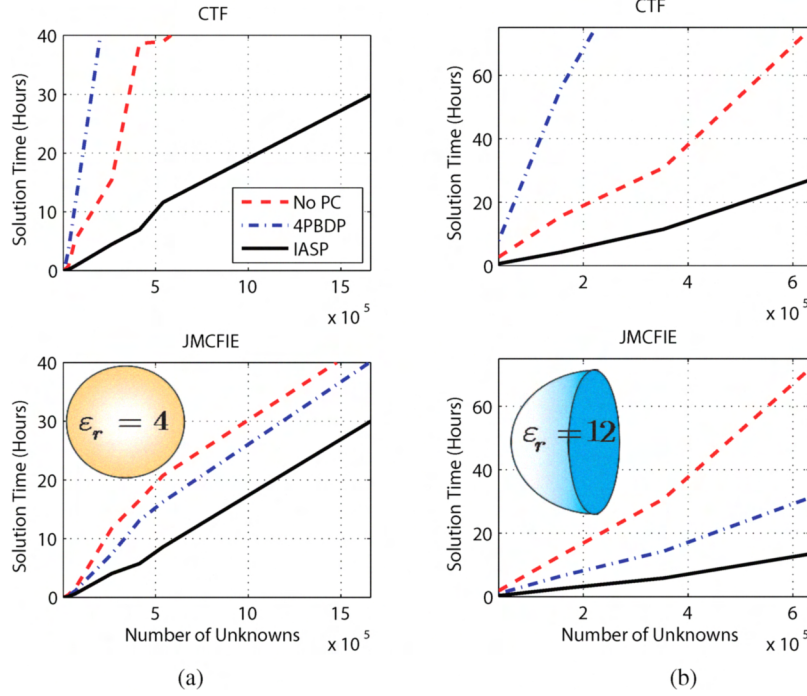


Figure 2: Number of iterations and the solution times for (a) the sphere problem and (b) the lens problem with CTF and JMCIE. The stopping criterion is set at 10^{-3} residual error.

In Fig. 3, we provide solution time versus norm of the residual plots for two real-life photonic crystal problems; one is a layered dielectric medium [6] and the other one is a slab material with inner holes [7]. These problems have relative dielectric constants of 4.8 and 11.56, respectively. The layered medium is solved at 300 MHz, has five layers with four meter walls, and involves 131,460 unknowns. The slab material is solved at 8.25 GHz, has 15×20 inner holes, and involves 162,420 unknowns. These problems cannot be solved with JMCIE using 4PBDP in 2,000 iterations. Also, with CTF formulation, GMRES cannot solve the problems without a preconditioner. On the other hand, using iterative ASP we have been able to solve these problems in approximately six hours with CTF and in two hours with JMCIE.

Conclusion

Integral-equation formulations for dielectrics have a wide range of application areas. However, iterative solutions of the resulting dense systems with MLFMA often suffers from slow convergence. We propose the iterative ASP preconditioner based on the Schur complement reduction technique to solve the near-field matrix system efficiently. The only cost of the preconditioner is the construction of a SAI for the (1,1) partition. Using this effective preconditioner, we obtain the CTF solutions of the sphere problems as fast as the JMCIE solutions. For other real-life problems, both CTF and JMCIE solutions can be obtained with ASP in modest times.

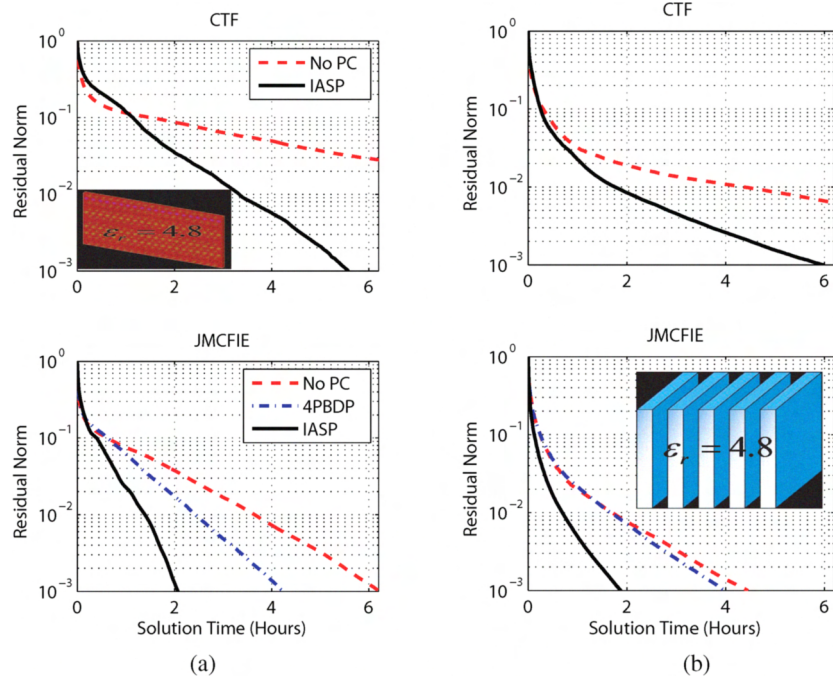


Figure 3: Number of iterations and the solution times for (a) the photonic crystal and (b) dielectric layers with CTF and JMCIE.

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